On the quantum mechanics of elliptic orbits in the Kepler problem (Preliminary study)

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I. KS TRANSFORMATION

The Kustaanheimo-Stiefel (KS) transformation was introduced [1] in order to remove the collision singularity of the two-body problem of celestial mechanics. By the convention of ref. [2], the transformation from the space $\mathbf{u} \in \mathbf{R}^4$ to original space $\mathbf{x} \in \mathbf{R}^3$ reads

$$x_1 = 2(u_1 u_3 - u_2 u_4); \quad x_2 = 2(u_1 u_4 + u_2 u_3); \quad x_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2. \tag{1}$$

The transformation implies the properties, see [1],

$$r \equiv u_1^2 + u_2^2 + u_3^2 + u_4^2 = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{2}$$

and any rotation $\mathbf{u}' = \mathrm{T}(\beta)\mathbf{u}$, with $0 \le \beta < 2\pi$, lets x_1, x_2, x_3, invariant:

$$\mathrm{T}(\beta) = \begin{pmatrix} \cos(\beta), & \sin(\beta), & 0, & 0 \\ -\sin(\beta), & \cos(\beta), & 0, & 0 \\ 0, & 0, & \cos(\beta), & -\sin(\beta) \\ 0, & 0, & \sin(\beta), & \cos(\beta) \end{pmatrix}. \tag{3}$$

By interpreting β as the forth component in x-space, $x_4 = \beta$, and using the polar representation $x_1 = r\sin(\theta)\cos(\varphi)$, $x_2 = r\sin(\theta)\sin(\varphi)$, $x_3 = r\cos(\theta)$, one obtains the following 1-1 map $(u_1, u_2, u_3, u_4) \leftrightarrow (r, \theta, \varphi, \beta)$ with $r \ge 0$, $0 \le \theta \le \pi$, $0 \le \varphi$, $\beta < 2\pi$:

$$\begin{array}{ll} u_1 = \sqrt{r}\cos(\theta/2)\cos(\varphi - \beta); & u_2 = \sqrt{r}\cos(\theta/2)\sin(\varphi - \beta); \\ u_3 = \sqrt{r}\sin(\theta/2)\cos(\beta); & u_4 = \sqrt{r}\sin(\theta/2)\sin(\beta). \end{array} \tag{4}$$

The corresponding functional determinant reads

$$\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 = \frac{\partial(u_1, u_2, u_3, u_4)}{\partial(r, \theta, \varphi, \beta)}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\beta = \frac{1}{8} r\sin(\theta)\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\beta = \frac{1}{8r}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}\beta, \tag{5}$$

which implies the following connection between volume elements

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\beta \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, f(x_1, x_2, x_3) \;&\equiv\; \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, f(x_1, x_2, x_3) \\ &= \frac{4}{\pi}\int r(u)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\, f(x_1(\mathbf{u}), x_2(\mathbf{u}), x_3(\mathbf{u})). \end{aligned} \tag{6}$$

In terms of the components x_1, x_2, x_3 and β, the inverse transformation reads

$$\begin{array}{l} u_1 = \frac{x_1\cos(\beta) + x_2\sin(\beta)}{\sqrt{2(r - x_3)}}; \qquad u_2 = \frac{x_2\cos(\beta) - x_1\sin(\beta)}{\sqrt{2(r - x_3)}}; \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}; \\ u_3 = \frac{1}{2}\sqrt{2(r - x_3)}\,\cos(\beta); \;\; u_4 = \frac{1}{2}\sqrt{2(r - x_3)}\,\sin(\beta) \end{array} \tag{7}$$

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As a check, when (4) is used and inserted on the right hand side of (1), then the polar representation of x_1, x_2, x_3 is reproduced, independently of β. From (7) one infers immediately

$$\tfrac{\partial u_1}{\partial \beta} = u_2,\ \ \tfrac{\partial u_2}{\partial \beta} = -u_1,\ \ \tfrac{\partial u_3}{\partial \beta} = -u_4,\ \ \tfrac{\partial u_4}{\partial \beta} = u_3, \tag{8}$$

which, in consistency with (1), implies the properties

$$\frac{\partial}{\partial \beta} x_k(\mathbf{u}) = 0, \quad k = 1,2,3; \quad \frac{\partial}{\partial \beta} r(\mathbf{u}) = 0. \tag{9}$$

To transform the Laplacian Δ_x of the one-particle Hamiltonian into u-space, we restrict, at first, to functions which depend via (1) on the u-components, briefly called xu space:

$$F(\mathbf{u}) := f(x_1(u), x_2(u), x_3(u)) \equiv f(\mathbf{x}), \quad f \in C^2. \tag{10}$$

Then, the following relation holds

$$\Delta_u F(u) \equiv \left[\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}\right] F(u) = 4r\Delta_x f(x). \tag{11}$$

This property is an immediate consequence of (1) which implies

$$\Delta_u \left[x_i(u)\right] = 0, \quad \text{and} \quad \sum_{j=1}^{4} \frac{\partial x_i(u)}{\partial u_j}\frac{\partial x_k(u)}{\partial u_j} = 4r\delta_{ik}, \quad i,k = 1,2,3. \tag{12}$$

where δ is the Kronecker symbol.

We write the stationary Schrödinger equation in configuration space $\mathbf{x} \in \mathbf{R}^3$ as

$$\mathrm{H}_x \psi(x) = E\psi(x) \quad \text{with} \quad \mathrm{H}_x = \left[-\frac{\hbar^2}{2\mu}\Delta_x - \frac{\kappa}{r}\right]\psi(x) = E\psi(x). \tag{13}$$

The coupling constant is specified as: $\kappa = GmM$ and $\kappa = q_e^2/(4\pi\epsilon_0)$ in the case of two gravitational masses m, M, and the hydrogen problem, respectively. Transforming (13) into xu space, using (12), we obtain

$$\mathrm{H}_{xu}\Psi(u) = E\Psi(u) \quad \text{with} \quad \mathrm{H}_{xu} = \left[-\frac{\hbar^2}{2\mu}\frac{1}{4r}\Delta_u - \frac{\kappa}{r}\right]\Psi(u) = E\Psi(u). \tag{14}$$

After multiplying this equation with $\mathrm{r} \equiv \mathbf{u}^2$, we get

$$\mathrm{H}_O\Psi(u) = \kappa\Psi(u), \quad \mathrm{H}_O = -\frac{\hbar^2}{8\mu}\Delta_u - E\,\mathbf{u}^2; \quad E < 0. \tag{15}$$

Thus, one has transformed the original Hamiltonian into the sum of four harmonic oscillators. However, the oscillators are coupled through xu space with $\Psi(\mathbf{u}) = \psi((\mathbf{x}(\mathbf{u}))$, which implies that $\Psi(u)$ does not separate into a product of four independent eigenfunctions of the harmonic oscillator, in general.

In the following we turn to the standard method of the transformation $\mathrm{H}_x \to \mathrm{H}_u$ with $\psi(\mathbf{u})$ unrestricted, see e.g. [2]. To this end a fourth differential δx_4 is introduced in addition to the complete differentials $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$:

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \\ \delta x_4 \end{pmatrix} \equiv A \begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_3 \\ \mathrm{d}u_4 \end{pmatrix}, \qquad A = \begin{pmatrix} 2u_3, & -2u_4, & 2u_1, & -2u_2 \\ 2u_4, & 2u_3, & 2u_2, & 2u_1 \\ 2u_1, & 2u_2, & -2u_3, & -2u_4 \\ u_2, & -u_1, & -u_4, & u_3 \end{pmatrix}. \tag{16}$$

The mutual orthogonality of the row vectors of the matrix A leads to the following transformed Laplacian and Hamiltonian $\mathrm{H}_Y \equiv \mathrm{H}_u$ [2] (the prefactor of Y is corrected here from 1/(4r) to 1/(4r^2)):

$$\Delta_x \;\to\; \frac{1}{4\mathrm{r}}\Delta_u - \frac{1}{4\mathrm{r}^2}\mathrm{Y}^2; \quad \mathrm{Y} = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}; \tag{17}$$

$$\mathrm{H}_Y \equiv \mathrm{H}_u \;=\; -\frac{\hbar^2}{2\mu}\left\{\frac{1}{4\mathrm{r}}\Delta_u - \frac{1}{4\mathrm{r}^2}\mathrm{Y}^2\right\} - \frac{\kappa}{\mathrm{r}}; \quad \mathrm{r} = u_1^2 + u_2^2 + u_3^2 + u_4^2. \tag{18}$$

The KS transformation (1) implies for all functions $f \in C^1$ the following property:

$$\mathrm{Y} f(\mathbf{x}(\mathbf{u})) \equiv 0. \tag{19}$$

This follows from

$$\mathrm{Y}\, x_i(u) \equiv 0, \quad i = 1,2,3, \tag{20}$$

which is an immediate consequence of the transformation (1). Property (19) is consistent with (14).

II. TWO DIFFERENT TIME PROPAGATORS

We define time propagation of an operator O by means of the commutator with the Hamiltonian

$$\dot{\mathrm{O}} = \frac{\mathbf{i}}{\hbar}\,[\mathrm{H}, \mathrm{O}]. \tag{21}$$

In the case of the oscillator Hamiltonian (15) this gives rise to a pseudo time, σ , of dimension s/m whereas the Hamitonian (18) generates true time t. In order to reveal the connection between σ and t, we determine the expectation values $\langle \dot{\mathbf{r}} \rangle$ for pseudo and real time, respectively. The expectation values are taken with respect to the coherent state of [2]

$$\psi(\mathbf{u}) = \exp[\mathbf{a}\cdot\mathbf{u} - \Gamma\,\mathbf{u}\cdot\mathbf{u}/2]; \quad \mathbf{a} = \sqrt{2}\gamma\,\boldsymbol{\alpha}; \quad (\mathbf{a})_k \equiv a_k = \mu_k + \mathbf{i}\,\nu_k,\ k = 1,2,3,4, \tag{22}$$

where

$$\gamma^2 = \Gamma; \quad \Gamma = \frac{4\mu\omega}{\hbar} \quad \text{and} \quad \omega^2 = \frac{(-E)}{2\mu}. \tag{23}$$

Furthermore, $E < 0$ is an energy eigenvalue of the stationary Schrödinger equation in original x-space, and α_k is the eigenvalue of the annihilation operator $a^{(k)}$ belonging to the k-th harmonic oscillator, $a^{(k)}|\alpha_k\rangle = \alpha_k|\alpha_k\rangle$. It is noted that ω has not the dimension of a frequency. In the space of H_O, normalization is obtained from the integral

$$1 = C_O^2 \int \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\,\psi^*\psi \quad \text{with} \quad C_O^2 = \frac{\Gamma^2}{\pi^2}\,\exp[-A^2/(4\Gamma)], \tag{24}$$

where

$$A^2 = (a_1 + a_1^*)^2 + (a_2 + a_2^*)^2 + (a_3 + a_3^*)^2 + (a_4 + a_4^*)^2. \tag{25}$$

In the case of H_Y we use the same state, but normalized within the metric (6):

$$1 = C_Y^2 \int \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\,r(u)\,\psi^*\psi \quad \text{with} \quad C_Y^2 = \frac{4\Gamma^4}{\pi^2}\,\frac{\exp[-A^2/(4\Gamma)]}{A^2 + 8\Gamma}. \tag{26}$$

Time dependence will be introduced indirectly via the pseudo time σ of H_O, which later on will be substituted by the true time t. With the σ-dependent phase,

$$a_k(\sigma) = a_k(0)\,\exp[-\mathbf{i}\,\omega\,\sigma], \tag{27}$$

the modified coherent state (22), $\psi \to \Psi$, fulfills the Schrödinger equation

$$-\frac{\hbar}{\mathbf{i}}\frac{\partial\Psi}{\partial\sigma} = \mathrm{H}_O\Psi, \quad \Psi = \frac{\Gamma}{\pi}\,\exp\left[-\mathbf{a}\cdot\mathbf{a}^*/(4\,\Gamma)\right]\,\exp\left[-\mathbf{i}\,\omega\,\sigma/2 - a^2(\sigma)/(4\,\Gamma)\right]\,\psi(\mathbf{u}) \tag{28}$$

with $\langle\Psi|\Psi\rangle_O = 1$. Up to a KS phase as defined in (3), the complex parameters $a_k(0)$, $k = 1,2,3,4$, will be fixed by prescribing expectation values of position and velocity at time $\sigma = t = 0$.

The velocity operators $\mathrm{dx}_i/\mathrm{d}\sigma$ and $\mathrm{dx}_i/\mathrm{d}t$, $i = 1,2,3$, are defined as follows:

$$\mathrm{w}_i \equiv \frac{\mathrm{dx}_i}{\mathrm{d}\sigma} = \frac{\mathbf{i}}{\hbar}\,[\mathrm{H_O},\,\mathrm{x}_i(\mathbf{u})]; \quad \mathrm{v}_i \equiv \frac{\mathrm{dx}_i}{\mathrm{d}t} = \frac{\mathbf{i}}{\hbar}\,[\mathrm{H_Y},\,\mathrm{x}_i(\mathbf{u})]; \quad i = 1,2,3. \tag{29}$$

The explicit expressions read:

$$\begin{aligned}
\mathrm{w}_1 &= -\mathbf{i}\frac{\hbar}{2\mu}\left[u_3\frac{\partial}{\partial u_1} - u_4\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_3} - u_2\frac{\partial}{\partial u_4}\right];\\
\mathrm{w}_2 &= -\mathbf{i}\frac{\hbar}{2\mu}\left[u_4\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_3} + u_1\frac{\partial}{\partial u_4}\right];\\
\mathrm{w}_3 &= -\mathbf{i}\frac{\hbar}{2\mu}\left[u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_4}\right]; \qquad (30)\\
\mathrm{v}_i &= \frac{1}{\mathrm{r}}\,\mathrm{w}_i; \quad \mathrm{r} = u_1^2 + u_2^2 + u_3^2 + u_4^2, \qquad (31)
\end{aligned}$$

where for the latter equation we applied (20) which implies that the commutators, $[\mathrm{x_i},\,\mathrm{Y}] = 0$, vanish.

Up to normalization factors, the u-integrals for $\langle \mathrm{w}_i\rangle$ and $\langle \mathrm{v}_i\rangle$ are identical, because in the latter integral the additional operator factor 1/r is compensated by the metric r. Using the normalization factors (24) and (26), the following connection results

$$\langle \mathrm{v}_i\rangle_Y = \frac{1}{S}\,\langle \mathrm{w}_i\rangle_O\,; \quad S = \left(\frac{C_O}{C_Y}\right)^2 = \frac{A^2 + 8\Gamma}{4\Gamma^2}. \qquad (32)$$

Through A^2 as defined in (25) and because of (27), S emerges as a σ-dependent re-scaling function, and because of the definitions given in (29), we get the following connection between real and pseudo time

$$\mathrm{d}t = S(\sigma)\,\mathrm{d}\sigma = \frac{A^2(\sigma) + 8\Gamma}{4\Gamma^2}\,\mathrm{d}\sigma \;=\; \frac{A^2(\sigma)}{4\Gamma^2}[1 + 2g(\sigma)]\,\mathrm{d}\sigma; \quad g(\sigma) = \frac{4\Gamma}{A^2(\sigma)}. \qquad (33)$$

The dimensionless function g will turn out as an extremely small magnitude in the case of the Kepler problem. Anticipating later results, g is of the order 10^{-70} for the Earth-Sun system.

Pseudo time σ was introduced in the path integral studies [3], [4] in the form $t \to \sigma$ with $\mathrm{d}t/\mathrm{d}\sigma = f(\mathbf{r}(\sigma))$ where f is some scalar function and $\mathbf{r}$ varies along a given path.

III. FIXING THE PARAMETER SPACE BY INITIAL CONDITIONS

A. Determination of the parameter vector a

We have in mind an orbit in the $[x_1,\, x_2]$ plane with $\langle x_3\rangle = 0$. The 8 real parameters μ_k, ν_k defined through $a_k(0) = \mu_k + \mathbf{i}\,\nu_k$, k=1,...4, by will be determined by identifying the mean values of position and velocity with the initial point $\mathbf{r}_0 = \{r_0, 0, 0\}$, $\mathbf{v}_0 = \{0, v_0, 0\}$. Anticipating elliptic orbits, the above point is one of the apsidal points, and implies that our coordinate axes coincide with the principal axes of the ellipse. The mean values are practically sharp in the case of the Earth-Sun system, see Section VII. As an additional condition we require, as was done in [2], that the mean value of the constraint operator Y vanishes. Altogether, this poses seven conditions for the eight real parameters. As a matter of fact, the complex parameters of the state (22) will be fixed up to a KS phase β according to (3).

The calculation of mean values by means of the coherent state (22), including mean square deviations, is outlined in Subsection A of Appendix A. Trust into the results can be based on their consistency with the elements of the elliptic orbit to be inferred, and also from an elementary approximation, as sketched in Appendix A, which gives the leading order of magnitudes up to quantum fluctuations of relative order g, see (33). In terms of the complex state parameters a_k, k=1,..4, the mean values of position, velocity and the constraint operator Y, defined in (17), read:

$$\begin{aligned}
\langle \mathrm{x}_1\rangle_Y &= \xi\,\left[(a_1 + a_1^*)(a_3 + a_3^*) - (a_2 + a_2^*)(a_4 + a_4^*)\right];\\
\langle \mathrm{x}_2\rangle_Y &= \xi\,\left[(a_1 + a_1^*)(a_4 + a_4^*) + (a_2 + a_2^*)(a_3 + a_3^*)\right];\\
\langle \mathrm{x}_3\rangle_Y &= \frac{1}{2}\,\xi\,\left[(a_1 + a_1^*)^2 + (a_2 + a_2^*)^2 - (a_3 + a_3^*)^2 - (a_4 + a_4^*)^2\right], \qquad (34)
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{v}_1\rangle_Y &= -\mathbf{i}\,\eta\,[a_1 a_3 - a_1^* a_3^* - a_2 a_4 + a_2^* a_4^*];\\
\langle \mathrm{v}_2\rangle_Y &= -\mathbf{i}\,\eta\,[a_2 a_3 - a_2^* a_3^* + a_1 a_4 - a_1^* a_4^*];\\
\langle \mathrm{v}_3\rangle_Y &= -\mathbf{i}\,\frac{1}{2}\,\eta\,\left[a_1^2 - (a_1^*)^2 + a_2^2 - (a_2^*)^2 - a_3^2 + (a_3^*)^2 - a_4^2 + (a_4^*)^2\right]; \qquad (35)
\end{aligned}$$

$$\langle \mathrm{Y} \rangle_O = \frac{1}{2\varGamma}\left[a_1 a_2^* - a_1^* a_2 + a_3^* a_4 - a_3 a_4^*\right], \tag{36}$$

where

$$\xi = \frac{1}{2\varGamma^2}\,\frac{1+4g}{1+2g}, \quad \eta = \frac{\hbar}{4\mu\,\varGamma\,S}. \tag{37}$$

It is convenient to represent μ_i and ν_i by means of planar polar coordinates as follows

$$\begin{aligned} &\mu_1 = \rho_{12}\cos(\varphi_{12});\;\; \mu_2 = \rho_{12}\sin(\varphi_{12});\;\; \mu_3 = \rho_{34}\cos(\varphi_{34});\;\; \mu_4 = \rho_{34}\sin(\varphi_{34});\\ &\nu_1 = R_{12}\cos(\phi_{12});\;\; \nu_2 = R_{12}\sin(\phi_{12});\;\; \nu_3 = R_{34}\cos(\phi_{34});\;\; \nu_4 = R_{34}\sin(\phi_{34}). \end{aligned} \tag{38}$$

Inserting (38) into (34) to (36) we obtain at time $\sigma = t = 0$:

$$\begin{aligned} r_0 = \langle \mathrm{x}_1(0)\rangle_Y &= 4\xi_0\,\rho_{12}\rho_{34}\cos\left(\varphi_{12}+\varphi_{34}\right);\\ 0 = \langle \mathrm{x}_2(0)\rangle_Y &= 4\xi_0\,\rho_{12}\rho_{34}\sin\left(\varphi_{12}+\varphi_{34}\right);\\ 0 = \langle \mathrm{x}_3(0)\rangle_Y &= 4\xi_0\,\left(\rho_{12}^2-\rho_{34}^2\right)/2;\\ 0 = \langle \mathrm{v}_1(0)\rangle_Y &= 2\eta_0\,\left[\rho_{12}R_{34}\cos\left(\varphi_{12}+\phi_{34}\right)+\rho_{34}R_{12}\cos\left(\varphi_{34}+\phi_{12}\right)\right];\\ v_0 = \langle \mathrm{v}_2(0)\rangle_Y &= 2\eta_0\,\left[\rho_{12}R_{34}\sin\left(\varphi_{12}+\phi_{34}\right)+\rho_{34}R_{12}\sin\left(\varphi_{34}+\phi_{12}\right)\right];\\ 0 = \langle \mathrm{v}_3(0)\rangle_Y &\equiv 2\eta_0\,\left[\rho_{12}R_{12}\cos\left(\varphi_{12}-\phi_{12}\right)-\rho_{34}R_{34}\cos\left(\varphi_{34}-\phi_{34}\right)\right];\\ 0 = \langle \mathrm{Y}\rangle_O &= -\frac{\mathbf{i}}{\varGamma}\left[R_{12}\rho_{12}\sin(\phi_{12}-\varphi_{12})-R_{34}\rho_{34}\sin(\phi_{34}-\varphi_{34})\right] \end{aligned} \tag{39}$$

where $\xi_0 = \xi(\sigma = 0)$ and $\eta_0 = \eta(\sigma = 0)$.

Because $\langle \mathrm{x}_3\rangle = 0$ we infer from (39) that

$$\rho_{12} = \rho_{34} \equiv \rho_0. \tag{40}$$

Furthermore, $\langle \mathrm{x}_2\rangle_Y = 0$ and $r_0 \equiv \langle \mathrm{x}_1\rangle_Y > 0$ imply $\varphi_{34} = -\varphi_{12}$, and $\langle \mathrm{v}_1\rangle = \langle \mathrm{v}_3\rangle = \langle \mathrm{Y}\rangle = 0$, together with the assumption $v_0 > 0$, give rise to the unique conditions

$$R_{34} = R_{12} \equiv \nu\,\rho_0; \quad \phi_{34} = -\varphi_{12} + \pi/2, \quad \phi_{12} = \varphi_{12} + \pi/2; \quad \varphi_{34} = -\varphi_{12}, \tag{41}$$

where instead of R_{12} we introduced the dimensionless parameter $\nu > 0$, which will turn out to be related to the eccentricity e. With this, the expressions (39) imply the following relations

$$\langle \mathrm{x}_1(0)\rangle_Y \equiv r_0 = 4\xi_0\,\rho_0^2; \quad v_0 = 4\eta_0\,\nu\,\rho_0^2. \tag{42}$$

The still open parameters ρ_0 and ν are thus fixed by the initial values r_0 and v_0, in particular

$$\rho_0^2 = \frac{r_0\varGamma^2}{2}\,\frac{1+2g_0}{1+4g_0}; \quad g_0 = \frac{4\varGamma}{A^2(0)}. \tag{43}$$

With the notation $\beta = \varphi_{12}$, the parameters μ_i, ν_i, as given in (38), now attain the special form

$$\begin{aligned} &\mu_1 = \rho_0\cos(\beta); \quad \mu_2 = \rho_0\sin(\beta); \quad \mu_3 = \rho_0\cos(\beta); \quad \mu_4 = -\rho_0\sin(\beta);\\ &\nu_1 = -\nu\,\rho_0\sin(\beta); \;\; \nu_2 = \nu\,\rho_0\cos(\beta); \;\; \nu_3 = \nu\,\rho_0\sin(\beta); \;\; \nu_4 = \nu\,\rho_0\cos(\beta). \end{aligned} \tag{44}$$

Time dependence is inferred from (27):

$$\{\mu_i,\,\nu_i\} \;\rightarrow\; \{M_i(\sigma),\,N_i(\sigma)\} = \{\mu_i\cos(\omega\,\sigma)+\nu_i\sin(\omega\,\sigma),\,\nu_i\cos(\omega\,\sigma)-\mu_i\sin(\omega\,\sigma)\}, \tag{45}$$

which with $a_k(\sigma) = M_k(\sigma) + \mathbf{i}\,N_k(\sigma)$ and in view of (44) leads to the assignments

$$\begin{aligned} &M_1 = \rho_0\left[\cos(\beta)\cos(\omega\,\sigma)-\nu\sin(\beta)\sin(\omega\,\sigma)\right], && N_1 = \rho_0\left[-\cos(\beta)\sin(\omega\,\sigma)-\nu\sin(\beta)\cos(\omega\,\sigma)\right],\\ &M_2 = \rho_0\left[\sin(\beta)\cos(\omega\,\sigma)+\nu\cos(\beta)\sin(\omega\,\sigma)\right], && N_2 = \rho_0\left[-\sin(\beta)\sin(\omega\,\sigma)+\nu\cos(\beta)\cos(\omega\,\sigma)\right],\\ &M_3 = \rho_0\left[\cos(\beta)\cos(\omega\,\sigma)+\nu\sin(\beta)\sin(\omega\,\sigma)\right], && N_3 = \rho_0\left[-\cos(\beta)\sin(\omega\,\sigma)+\nu\sin(\beta)\cos(\omega\,\sigma)\right],\\ &M_4 = \rho_0\left[-\sin(\beta)\cos(\omega\,\sigma)+\nu\cos(\beta)\sin(\omega\,\sigma)\right], && N_4 = \rho_0\left[\sin(\beta)\sin(\omega\,\sigma)+\nu\cos(\beta)\cos(\omega\,\sigma)\right]. \end{aligned} \tag{46}$$

The open phase β is related to the rotation invariance (3) of the KS-transformation. To see this, we specify the parameter vector $\mathbf{a}$ of the wave function (22) by the initial conditions with the aid of (46). The relevant exponent of ψ then reads

$$\begin{aligned}\mathbf{a}(\beta)\cdot\mathbf{u} &= \rho_0\,\exp[-\mathbf{i}\,\omega\,\sigma]\,\{(u_1+u_3)\cos(\beta)+(u_2-u_4)\sin(\beta)\\ &+\ \mathbf{i}\,\nu\,[(u_2+u_4)\cos(\beta)-(u_1-u_3)\sin(\beta)]\}\,. \end{aligned} \tag{47}$$

One observes the property

$$\begin{aligned}\mathbf{a}(\beta)\cdot\mathbf{u} &= \mathbf{a}(\mathbf{0})\cdot\mathbf{u}', \quad \mathbf{u}'=[\mathrm{T}(\beta)\mathbf{u}];\\ \psi_\beta(\mathbf{u}) &= \psi_0(\mathbf{u}'). \end{aligned} \tag{48}$$

This reflects the invariance of the coordinates $x_i(\mathbf{u})$, i=1,2,3, with respect to transformations $\mathbf{u}\to\mathrm{T}(\beta)\mathbf{u}$.

B. Probability density $P(x, y)$

The expectation values we calculate turn out to be independent of the KS phase β. The same is true for the probability density, $P(x, y)$, to find the planet at the point (x, y). This is calculated in Appendix B, approximately up to terms of relative order g_0, with the result:

$$\begin{aligned}P(x,y) &= C_Y^2\int_{\mathbf{R}^4}\mathrm{d}\mathbf{u}\,u^2\,\delta(x-x_1(\mathbf{u}))\delta(y-x_2(\mathbf{u}))\,|\psi(\mathbf{u})|^2\\ &= \frac{1}{4\pi\,\zeta}\,\exp\left[-\frac{(x-\langle x\rangle)^2}{4\zeta}-\frac{(y-\langle y\rangle)^2}{4\zeta}\right](1+\mathcal{O}(g_0));\quad \zeta=\frac{r_0 Z(\sigma)}{2\varGamma(\sigma)}, \end{aligned} \tag{49}$$

where the mean position coordinates $\langle x\rangle$, $\langle y\rangle$ and the function Z are specified further below in (51) and (54); and $\varGamma(\sigma)$ describes dispersion, as is approximately determined in Appendix D with the result (D.19). It is seen, that the relative mean square root deviation $\Delta(x,y)/r_0=\sqrt{4\zeta}/r_0\approx\sqrt{g_0}$, which in the case of the Earth-Sun system is of the order of magnitude 10^{-37}, see Subsection VI. E. below. The width $\Delta(x, y)$ is timely modulated along the elliptic orbit through the factor $\varGamma(0)\,Z/\varGamma(\Upsilon)=2\,[(1+e\cos(\Upsilon))/(1+e)]^{5/4}$, where e denotes the eccentricity and Υ the eccentric anomaly, see Subsections VI.A. and C.

IV. TIME DEPENDENT MEAN VALUES

When (49) is applied to the mean values of the phase space variables (34) and (35), it is straightforward to show that the mean orbit stays in the $[x_1,\, x_2]$ plane, i.e.

$$0=\langle \mathrm{x}_3(\sigma)\rangle_Y;\quad 0=\langle \mathrm{v}_3(\sigma)\rangle_Y\quad \text{for}\quad \sigma\geq 0. \tag{50}$$

Furthermore, we find

$$\begin{aligned}x(\sigma)\equiv\langle \mathrm{x}_1(\sigma)\rangle_Y &= \frac{\rho_0^2}{\varGamma^2}\left[1-\nu^2+(1+\nu^2)\,\cos(2\omega\,\sigma)\right]\frac{1+4g(\sigma)}{1+2g(\sigma)};\\ y(\sigma)\equiv\langle \mathrm{x}_2(\sigma)\rangle_Y &= \frac{2\nu\rho_0^2}{\varGamma^2}\,\sin(2\sigma\omega)\,\frac{1+4g(\sigma)}{1+2g(\sigma)}. \end{aligned} \tag{51}$$

$$\begin{aligned}v_x\equiv\langle \mathrm{v}_1(\sigma)\rangle_Y &= -\frac{\hbar(1+\nu^2)\rho_0^2}{2\mu\varGamma}\,\frac{\sin(2\omega\,\sigma)}{S(\sigma)};\\ v_y\equiv\langle \mathrm{v}_2(\sigma)\rangle_Y &= \frac{\hbar\nu\rho_0^2}{\mu\varGamma}\,\frac{\cos(2\omega\,\sigma)}{S(\sigma)}. \end{aligned} \tag{52}$$

For the amplitude function A^2, as defined in (25), and the scaling function S, defined in (33), one obtains with the aid of (43)

$$A^2(\sigma) = 4\rho_0^2\,Z(\sigma), \tag{53}$$

$$Z(\sigma) = 1+\nu^2+(1-\nu^2)\,\cos(2\omega\,\sigma); \tag{54}$$

$$S(\sigma) = \frac{1}{2} r_0 \, Z(\sigma) \, [1 + 2g(\sigma)] \, \frac{1 + 2g_0}{1 + 4g_0}; \tag{55}$$

$$\begin{aligned} \langle \mathrm{r} \rangle_Y \equiv \langle u_1^2 + u_2^2 + u_3^2 + u_4^2 \rangle_Y &= \frac{\rho_0^2}{\Gamma^2} \, Z(\sigma) \, \frac{1 + 6g(\sigma) + 6g^2(\sigma)}{1 + 2g(\sigma)} \\ &= \frac{1}{2} r_0 \, Z(\sigma) \, \frac{1 + 6g(\sigma) + 6g^2(\sigma)}{1 + 2g(\sigma)} \, \frac{1 + 2g_0}{1 + 4g_0}. \end{aligned} \tag{56}$$

By the definition of the fluctuation function g, see (33) and (43), the constant $g_0 \equiv g(0)$ is implicitly determined:

$$g_0 \equiv \frac{4\Gamma}{A^2(0)} = \frac{1}{r_0 \Gamma} \, \frac{1 + 4g_0}{1 + 2g_0}. \tag{57}$$

Obviously, the inequality $g_0 < 2/(r_0 \Gamma)$ holds, independently of the magnitude of $r_0 \Gamma$. Because $\mathrm{d}\, t/\mathrm{d}\, \sigma \equiv S > 0$, the real time is uniquely determined by pseudo time, and the vanishing of $\langle x_3(\sigma) \rangle = 0$ and $\langle v_3(\sigma) \rangle = 0$ hold true also for real time $t \geq 0$.

V. ELEMENTS OF ELLIPTIC ORBIT

In the following we anticipate an elliptic motion and work out its elements. By our choice of initial conditions, the zero time, $t = \sigma = 0$, will turn out to refer to aphelion.

A. Semimajor axis and eccentricity

In this and the next subsection we will neglect quantum corrections of relative order g_0. From (56) we obtain with the aid of (54) and (43) the following time dependent distance:

$$r(\sigma) \overset{g_0 \to 0}{=} \frac{1}{2} r_0 \left[1 + \nu^2 + (1 - \nu^2) \cos(2\omega \, \sigma) \right]. \tag{58}$$

If a and e denote the length of the semimajor axis and the eccentricity, respectively, then by definition $r_{max} = a(1 + e)$ and $r_{min} = a(1 - e)$, and from (58) we infer for $\nu^2 > 1$

$$a(1 - e) \equiv \min_\sigma \, r(\sigma) = r_0 \quad \text{and} \quad a(1 + e) \equiv \max_\sigma \, r(\sigma) = \nu^2 \, r_0, \tag{59}$$

which, by division, gives rise to

$$\nu^2 = \frac{1 + e}{1 - e} \quad \text{or} \quad e = \frac{\nu^2 - 1}{\nu^2 + 1}, \quad \text{and} \quad a = \frac{r_0}{1 - e}. \tag{60}$$

Eq.(59) implies that the initial position $(r_0, 0, 0)$ is at perihelion with $r_0 = a(1 - e)$ as it must be.

B. Choosing the energy parameter E

We compare the results (52) for the velocity with the following formula for the kinetic energy E_{kin} as given by Eq. (4.36) in [5] (here slightly adapted):

$$E_{kin}(\sigma) \equiv \frac{1}{2} \mu V^2 = \frac{\kappa}{2} \left(\frac{2}{r(\sigma)} - \frac{1}{a} \right). \tag{61}$$

With the aid of (52), (55), (60), (85) and (43) we obtain

$$\tilde{E} \equiv \frac{\mu}{2} (v_x^2 + v_y^2) - \frac{\kappa}{r(\sigma)} \overset{g_0 \to 0}{=} \frac{\hbar^2 \Gamma^2}{8\mu} \, \frac{1 + e \cos(2\omega \, \sigma)}{1 - e \cos(2\omega \, \sigma)} - \frac{\kappa}{a} \, \frac{1}{1 - e \cos(2\omega \, \sigma)}, \tag{62}$$

and this is independent of σ if we one sets

$$\kappa/a = \hbar^2 \Gamma^2/(4\mu). \tag{63}$$

TABLE I. The physical constants are: μ, $M_\odot$ masses of Earth, Sun; G, a gravitational, astronomical constant.

$\mu = 5.979 \times 10^{24}$ kg, $M_\odot = 1.991 \times 10^{30}$ kg,	$G = 6.673 \times 10^{-11}$ kg^{-1}m^{3}s^{-2},
$a = 1.49596 \times 10^{11}$ m,	$\hbar = 1.0546 \times 10^{-34}$ kg m^{2}s^{-1}.

Now, after substituting in (62) $\Gamma^2 = 8\mu(-E)/\hbar^2$, which follows from (23), formula (62) leads to the relation:

$$\tilde{E} = E = -\frac{\kappa}{2a} = -(1-e)\frac{\kappa}{2r_0}, \quad 0 \le e < 1. \tag{64}$$

This result is equivalent to (61), and a well known property of a Kepler ellipse, see e.g. Eq. (3.52) of [6]. A parabolic orbit corresponds to $E = 0$ with eccentricity $e = 1$.

We also find that the mean value of the Hamiltonian,

$$\langle \mathrm{H}_Y \rangle = E[1 + \mathcal{O}(g_0)], \tag{65}$$

is constant in time up to quantum corrections of order g_0, see Appendix C.

C. Pseudo time and Kepler's equation

With the aid of (53), (54), (43) and (60), the scaling function $S(\sigma)$, which according to (33) connects pseudo time σ with real time t, can be written as follows

$$S(\sigma) \overset{g_0 \to 0}{=} a\,[1 - e\cos(2\omega\,\sigma)]\,. \tag{66}$$

We integrate (33) with initial condition $t_0 = \sigma_0 = 0$ and multiply the resulting equation with $2\omega/a$ to obtain

$$\frac{2\omega}{a}\,t = [2\omega\,\sigma - e\,\sin(2\omega\,\sigma)]\,. \tag{67}$$

Now, we identify $M = (2\omega/a)\,t$ with the mean anomaly, and $\Upsilon = 2\omega\,\sigma$ with the eccentric anomaly to write

$$M = [\Upsilon - e\sin(\Upsilon)]\,. \tag{68}$$

with

$$M = \frac{2\omega}{a}\,t \quad \text{and} \quad \Upsilon = 2\omega\,\sigma. \tag{69}$$

This is Kepler's equation [7], see e.g. Eq.(4.60) in [5].

D. Period of elliptic motion

In this subsection we further neglect quantum corrections of relative order g_0. The period $\Upsilon_0 = 2\pi$ implies the time period T_0:

$$\frac{2\omega}{a}\,T_0 = 2\pi, \quad \text{or} \quad T_0 = a\,\pi\sqrt{\frac{2\mu}{(-E)}} = 2\pi\,a\sqrt{\frac{a}{GM_\odot}}, \tag{70}$$

where we used (64), (23) and $\kappa = G\mu M_\odot$. For a numerical value we adopt the physical constants of Tab.I to obtain

$$T_0 = 3.1548 \times 10^7\,s = 0.9997\,\text{tropical year}, \tag{71}$$

where the tropical year has 365.2452 days, see e.g. p. 508 [5].

E. Analyzing the g function

The dimensionless function g, defined in (33), characterizes the disturbance of the classical orbit by quantum corrections. With the aid of (53), (54), (43), (57) and (60) we can write in terms of the eccentric anomaly Υ

$$g(\Upsilon) \equiv \frac{4\Gamma}{A^2} = \frac{1}{r_0\Gamma}\,\frac{1-e}{1-e\cos(\Upsilon)}\,\frac{1+4g_0}{1+2g_0}, \tag{72}$$

which implies the following bounds for g:

$$g(\Upsilon) \leq g_0; \quad g_0 \equiv g(0) = \frac{\theta}{r_0\Gamma}, \quad 1 \leq \theta \leq 2. \tag{73}$$

In the parabolic limit with $E = 0$ $(e = 1)$, Γ behaves proportional to $\sqrt{(-E)}$, which makes g_0 singular.

The quantum number g_0 can be expressed by the quotient of two time parameters as follows, using (70) and (23):

$$g_0 \approx \frac{1}{r_0\Gamma} \approx \frac{\hbar}{2\mu a^2}\,\frac{\pi a}{\omega} = \frac{T_0}{T_a}; \quad T_a = \frac{2\mu a^2}{\hbar}, \tag{74}$$

where T_a can be interpreted as the spreading time of a Gaussian wave packed, which initially is confined to a box of width a, whereas the orbit period T_0 reflects the effect of the attractive potential. Numerically, we obtain for the Earth-Sun system (ES)

$$g_0^{(ES)} \approx 2\times 10^{-75}. \tag{75}$$

In the case of the hydrogen atom in a state with principal quantum number n, the energy E and Bohr radius a_B read in international units

$$E_n = -\frac{q_e^4\mu}{32\pi^2\epsilon_0^2\hbar^2}\,\frac{1}{n^2} \quad \text{and} \quad a_B = \frac{4\pi\epsilon_0\hbar^2}{\mu q_e^2}, \tag{76}$$

where q_e, μ denote electron charge and mass, respectively, and ϵ_0 is the electric constant. In the expression of $g_0 = \theta/(r_0\Gamma)$ we set $r_0 = n^2\, a_B$ and $\Gamma = 4\mu\sqrt{(-E_n)/(2\mu)}$; the indeterminacy parameter θ depends on the energy E_n, and is easily determined by solving (57) for g_0. One obtains

$$g_0^H = \frac{\theta_n}{2\,n}, \quad 1 \leq \theta_n \leq 2. \tag{77}$$

In the ground state $g_0^H \approx 0.81$, whereas in an experimentally accessible Rydberg state with mean quantum number n=72, see [8], [9], the quantum number $g_0^H \approx 0.007$.

F. Hodograph

As was shown by J. C. Maxwell in [10], the hodograph of the Kepler motion is a circle (The hodograph is the line traced out by the end points of the velocity vectors with a fixed origin). By relating the hodograph to the elliptic orbit by means of Kepler's laws, Maxwell derived the law of gravitation. Later on, R. Feynman apparently used Maxwell's design, Fig.16 in [10], in one of his "Lost Lectures" to construct the elliptic orbit from the hodograph in an elementary geometric way; for recent discussions see [11],[12],[**?**]. In the following we apply our results for a reconstruction of Maxwell's design.

We write in terms of the eccentric anomaly $\Upsilon = 2\omega\,\sigma$, neglecting terms of relative order g_0,

$$S \overset{g_0\to 0}{=} \frac{r_0}{2} Z(\Upsilon) = a[1-e\cos(\Upsilon)], \tag{78}$$

where we used (55), (54) and (60). By means of (52), (43), (60) and (23), we obtain

$$V_1 \equiv v_x = -2\,\omega\frac{\sin(\Upsilon)}{1-e\cos(\Upsilon)}; \quad V_2 \equiv v_y = 2\omega\sqrt{1-e^2}\frac{\cos(\Upsilon)}{1-e\cos(\Upsilon)}. \tag{79}$$

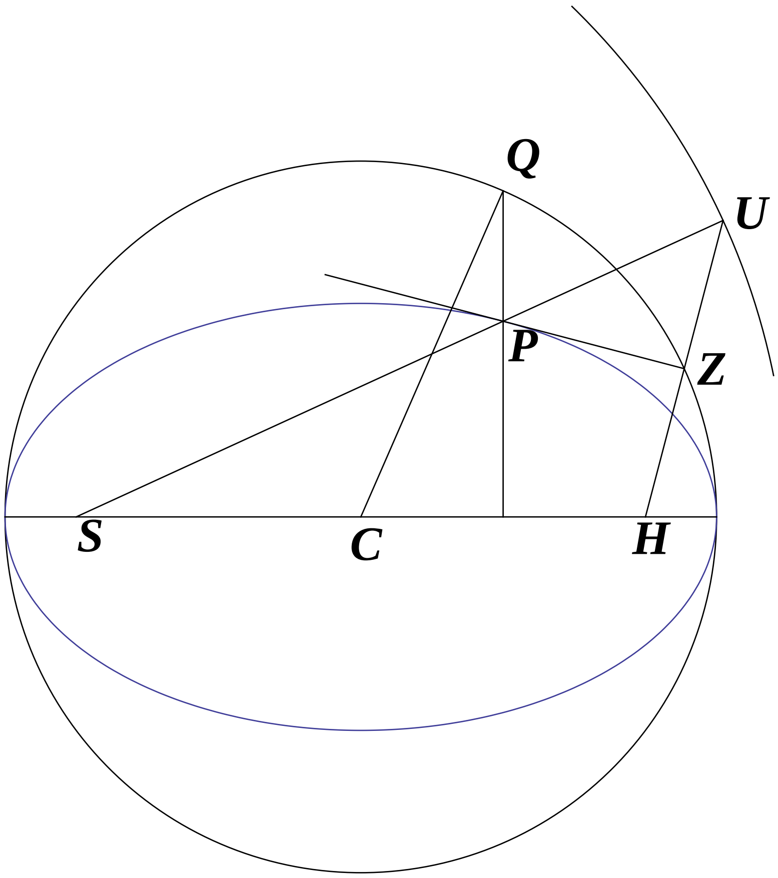

Figure 1: Modified version of J.C. Maxwell's Fig.16 in [10]. The hodograph with center in the focus S has radius $2a$, a semimajor axis; $\overrightarrow{SP}$ is the vector of the planet, the vector $\overrightarrow{HU}$ points from the focus H to the hodograph and is perpendicular to the velocity (tangent) at P with lengths HZ=ZU; the length HU is proportional to the speed at P. The angle $\widehat{HCQ}$ is the eccentric anomaly which here is zero at aphelion.

We now shift the coordinate origin in velocity space to the point $(0,\, V_0)$, where

$$V_0 = \frac{2\,e}{\sqrt{1-e^2}}\,\omega, \tag{80}$$

and find after elementary simplifications that

$$V_1^2 + (V_2 - V_0)^2 = \frac{4\omega^2}{1-e^2} = \text{constant} \tag{81}$$

This shows that the end points of the velocity vector $\{V_1(\Upsilon),\, V_2(\Upsilon)\}$ lie on a circle.

In order to combine the hodograph with the Kepler ellipse in the same plot, Maxwell mapped the velocities to space vectors: $\mathbf{V} \to \mathbf{R}$ by scaling with the constant of motion f:

$$\mathbf{V} = f\,\mathbf{R}; \quad f = \frac{L_z}{2\mu a^2(1-e^2)} = \frac{\omega}{a\sqrt{1-e^2}}, \tag{82}$$

where for the last equation we substituted in the expression of angular momentum $L_z \equiv \mu\, r_0\, v_0$, the speed v_0 by means of (42) and used (37) together with $S(0) = r_0$, (23) and (60). With this rescaling, the hodograph (81) is a circle with radius $2a$:

$$R_1^2 + (R_2 - R_0)^2 = (2a)^2; \quad R_0 = 2ea. \tag{83}$$

How to orient orbit and hodograph to each other? The ellipse, which arises from $\{x(\sigma),\, y(\sigma)\}$ according to (51) (with $g_0 \to 0$), has its focuses on the x-axis. In order that the analysis works, the origin of the hodograph with radius $(2a)$ has to coincide with the Sun in one focus, and the origin of the velocity vectors, so far at $(0, 2ea)$, in the other focus. To this end we rotate the velocity vectors clockwise by $\pi/2$: $\{R_1,\, R_2\} \to \{R_2,\, -R_1\}$, so that the rotated vector is perpendicular to the velocity. Simultaneously, we must shift $\{0,\, R_0\} \to \{R_0,\, 0\}$, with $R_0 = 2ea$, which brings the origin of the vectors $\mathbf{R}$ into to other focus. The situation is illustrated in Fig.1, which is a rotated version of Fig. 16 in [10], and additionally reminds of the definition of the eccentric anomaly.

VI. FLUCTUATIONS

In Eq. (3.41) of reference [2], the mean square deviations (msq) of the position components are reported, calculated in the metric of H_O. With the symbol Γ of this paper, the result of Gerry [2] reads (we corrected the factor 1/2 into 2)

$$(\Delta x_i)^2 = \frac{2}{\Gamma}\,\left[\langle x_1(\sigma)\rangle^2 + \langle x_2(\sigma)\rangle^2 + \langle x_3(\sigma)\rangle^2\right]^{1/2}. \tag{84}$$

With the aid of (34) and (37), and using (72), we can write

$$(\Delta x_i)^2 = \frac{A^2}{2\Gamma^3}[1+\mathcal{O}(g)] = \frac{2a}{\Gamma}\,\frac{1}{a\Gamma\, g}\,[1+\mathcal{O}(g)] = \frac{2a}{\Gamma}\,[1-e\cos(\Upsilon)]\,[1+\mathcal{O}(g)] \tag{85}$$

which is in agreement with the result (A.37) of this paper. The result (85) tells, that the msq of position is largest at aphelion of the orbit. If the eccentricity e is small, the timely modulation (via Kepler's equation (68)) is weak. For a numerical value we express Γ according to (23) and set

$$\omega = \sqrt{\frac{(-E)}{2\mu}} = \sqrt{\frac{\kappa}{4a\mu}};\quad \kappa = G\mu M_\odot\,. \tag{86}$$

After inserting the physical constants of Tab.I, we obtain for the Earth-Sun system

$$\Delta x_i \approx 9.4\times 10^{-27}\,\sqrt{1-e\cos(\Upsilon)}\,\mathrm{m}. \tag{87}$$

The msq of the velocity is given by (A.46) (it is not calculated in [2]):

$$(\Delta v_i)^2 = \frac{4\omega^2}{a\Gamma}\,\frac{1}{(1-e\cos(\Upsilon))^2}\,[1+\mathcal{O}(g)]. \tag{88}$$

With the physical constants of the Earth-Sun system we obtain

$$\Delta v_i \approx 1.3\times 10^{-33}\,\frac{1}{1-e\cos(\Upsilon)}\,[1+\mathcal{O}(g)]\,\mathrm{m/s}. \tag{89}$$

The msqr (87) and (89) are beyond the possibility of detection.

We check the uncertainty relation. From (85) and (88) we derive, up to terms of relative order g,

$$\Delta x_i \Delta p_i \equiv \mu\,\Delta x_i \Delta v_i = \sqrt{2}\,\frac{2\mu\omega}{\Gamma}\,[1-e\cos(\Upsilon)]^{-1/2} = \frac{\hbar}{\sqrt{2}}\,[1-e\cos(\Upsilon)]^{-1/2}. \tag{90}$$

This inequality fulfills the necessary condition $\Delta x_i \Delta p_i \geq \hbar/2$, because

$$[2\,(1-e\cos(\Upsilon))]^{-1/2} \geq 1/2. \tag{91}$$

Up to correction terms of relative order g, at aphelion with $\Upsilon = 0$ and for limiting eccentricity $e \to 1$, the uncertainty product in (90) has its lowest possible value, which further illustrates the quality of the state $\psi(\mathbf{u})$ defined in (22).

VII. APPENDIX

A. Mean values

1. Order of magnitudes

In this appendix we first sketch elementary estimates by taking into account the narrow width $1/\sqrt{\Gamma}$ of the probability density related to $\psi(\mathbf{u})$. The estimates should make plausible the exact results to follow. The mean values of position, velocity, angular momentum, and the corresponding mean square deviations have the structure

$$\langle ... \rangle_Y = \int \mathrm{d}^4 u\, f_n(\mathbf{u})\, P(\mathbf{u}), \quad P(\mathbf{u}) = C_Y^2\, |\psi(\mathbf{u})|^2, \tag{A.1}$$

where f_n is polynomial of order n, which includes the metric function $r \equiv u_1^2 + ...u_4^2$, and in some cases a factor $1/r$. Using the wave function (22), we shift the variables $\mathbf{u} \to \mathbf{U}$ with

$$\mathbf{U} = \mathbf{u} - \mathbf{D}, \quad D_i = (a_i + a_i^*)/(2\Gamma), \quad i = 1, ..., 4, \tag{A.2}$$

and obtain using (26):

$$P(\mathbf{u}) = \tilde{P}(\mathbf{U}) = \frac{4\Gamma^4}{\pi^2(A^2 + 8\Gamma)}\, \exp[-\Gamma U^2]. \tag{A.3}$$

Taylor expansion of f_n at $\mathbf{u} = \mathbf{D}$ leads to

$$f_n(\mathbf{u}) = f_n(\mathbf{D}) + f_n(\mathbf{D})_k U_k + \frac{1}{2}\, f_n(\mathbf{D})_{kl} U_k U_l + \dots \tag{A.4}$$

where the subscripts k, l of f_n denote partial derivatives with respect to u_k, u_l and summation convention is adopted. By symmetry the integrals of uneven polynomials in $\mathbf{U}$ vanish. Thus, we obtain

$$\int \mathrm{d}^4 u\, f_n(\mathbf{u})\, P(\mathbf{u}) = F_0 + F_2 + \dots \tag{A.5}$$

with

$$F_0 = f_n(\mathbf{D}) \int \mathrm{d}U_1 ... \mathrm{d}U_4\, \tilde{P}(\mathbf{U}) = \frac{4\Gamma^2}{A^2 + 8\Gamma}\, f_n(\mathbf{D}); \tag{A.6}$$

$$F_2 = \sum_{k=1}^{4} K_k f_n(\mathbf{D})_{kk}, \quad K_k = \frac{1}{2} \int \mathrm{d}U_1 ... \mathrm{d}U_4\, U_k^2 \tilde{P}(\mathbf{U}) = \frac{\Gamma}{A^2 + 8\Gamma}. \tag{A.7}$$

As compared to F_0, essentially, in F_2 the factor u_k^2 is replaced by U_k^2, where the first is of order D_k^2 and the latter of order $1/\Gamma$. Thus,

$$F_2/F_0 \approx \frac{1}{\Gamma \sum_{k=1}^4 D_k^2} = \frac{4\Gamma}{A^2} \approx g_0 \tag{A.8}$$

has the order of magnitude of the quantum fluctuation number g_0. To leading order, the mean values are therefore given by F_0.

2. Mean values to leading order

As to mean position, the polynomial f_n reads

$$f_n \to f_4 = r\, x_i(\mathbf{u}) \tag{A.9}$$

with x_i given in (1). In view of (A.6), we obtain

$$\langle x_i\rangle_Y \approx \frac{4\Gamma^2}{A^2+8\Gamma}\sum_{k=1}^{4} D_k^2\, x_i(\mathbf{D}) = \frac{A^2}{A^2+8\Gamma}x_i(\mathbf{D}) = \frac{1}{1+2g}x_i(\mathbf{D}). \tag{A.10}$$

Up to quantum corrections of relative order g, the latter result is equal to the exact result given in (34) and (37).

The real time velocity operators are defined in (31) and (30). In the integrand for the mean values, due to the metric factor, the operator $\mathrm{r}\,\mathbf{v} = \mathbf{w}$. We exemplarily determine the first component $V_1 := \langle \mathrm{v}_1\rangle_Y$ by using the differential operator (30):

$$V_1 = C\, C_Y^2 \int \mathrm{d}u_1 ... \mathrm{d}u_4\, \psi^* \left(u_3\frac{\partial}{\partial u_1} - u_4\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_3} - u_2\frac{\partial}{\partial u_4}\right)\psi, \quad C = -\mathbf{i}\frac{\hbar}{2\mu}. \tag{A.11}$$

After applying the operator to ψ, we infer the polynomial

$$f_n \to f_2 = C\left(a_3u_1 - a_4u_2 + a_1u_3 - a_2u_4 - 2\Gamma(u_1u_3 - u_2u_4)\right) \tag{A.12}$$

With the aid of (A.6), one easily verifies that

$$V_1 = f_2(\mathbf{D})\frac{4\Gamma^2}{A^2+8\Gamma} \tag{A.13}$$

gives the result stated by (35) and (37). The leading order is exact because f_n, here, is a second order polynomial which leads to vanishing higher order terms in (A.5).

We estimate the mean value of the angular momentum $\mathbf{l} = \mu\,\mathbf{x}\times\mathbf{v}$. The mean value of a typical constituent is

$$\mu\langle \mathrm{x}_2\mathrm{v}_1\rangle_Y \to F_0 = \mu\, x_2(\mathbf{D}) f_2\, \frac{4\Gamma^2}{A^2+8\Gamma} = (1+2g)\langle \mathrm{x}_2\rangle_Y\, \langle \mathrm{v}_1\rangle_Y, \tag{A.14}$$

with f_2 given in (A.12), and we made use of (A.10) and (A.13). We can therefore write

$$\langle \mathbf{l}\rangle_Y = \mu\, \left[\langle \mathbf{x}\rangle_Y \times \langle \mathbf{v}\rangle_Y\right]\left[1+\mathcal{O}(g)\right]. \tag{A.15}$$

If Z_i denotes an operator component x_i, or v_i, or l_i, then it will turn out that

$$\langle \mathrm{Z}_i^2\rangle_Y = \langle \mathrm{Z}_i\rangle_Y^2\left[1+\mathcal{O}(g)\right]. \tag{A.16}$$

To show this, we begin with $\langle \mathrm{x}_i^2\rangle_Y$, which gives rise to the polynomial

$$f_n \to f_6 = (u_1^2 + ... u_4^2)\, x_i^2(\mathbf{u}) \quad\longrightarrow\quad F_0 = \frac{1}{1+2g}\, x_i^2(\mathbf{D}), \tag{A.17}$$

which according to (A.10) is equal to $\left[\langle \mathrm{x}_i\rangle_Y\right]^2$ up to relative order g. As to the first velocity component, in the integrand

$$\langle \mathrm{v}_1^2\rangle_Y = C_Y^2 \int \mathrm{d}u_1 ... \mathrm{d}u_4\, \psi^*\, \mathrm{w}_1 \frac{1}{\mathrm{r}}\, \mathrm{w}_1\, \psi \tag{A.18}$$

we shift the first operator w_1 to ψ^* by partial integration, which produces a minus sign, and obtain with f_2 from (A.12)

$$f_n \to f_4 = \frac{1}{r}\, f_2^* f_2. \tag{A.19}$$

Because of $f_2(\mathbf{D}) = f_2^*(\mathbf{D})$, one obtains

$$F_0 = \frac{4\Gamma^2}{A^2+8\Gamma}\,\frac{1}{D^2} f_2^2(\mathrm{D}) = \frac{16\Gamma^4}{A^2(A^2+8\Gamma}\, f_2^2(\mathbf{D}). \tag{A.20}$$

Since $1/A^2 = (1+2g)/(A^2+8\Gamma)$, the above result is equal to $\langle \mathrm{v}_1\rangle^2\,[1+\mathcal{O}(g)]$. To examine $\langle \mathrm{l}_i^2\rangle_Y$, it is sufficient to consider a general constituent $Q = \mu^2\,\langle \mathrm{x}_i\,\mathrm{v}_k\,\mathrm{x}_m\,\mathrm{v}_n\,\rangle_Y$ with $i \neq k$ and $m \neq n$. In the integral

$$Q = \mu^2\, C_Y^2 \int \mathrm{d}u_1 ... \mathrm{d}u_4\, \psi^* \mathrm{x}_i\, \mathrm{w}_k\, \mathrm{x}_m\, \frac{1}{\mathrm{r}} \mathrm{w}_n \psi \tag{A.21}$$

we shift the operator w_k to ψ^* by partial integration and use f_2 from (A.12) with the proper index markers to obtain, using (A.10) and (A.13),

$$\begin{aligned} Q \to F_0 &= \mu^2\, \frac{4\Gamma^2}{A^2+8\Gamma}\,(f_2^*)_k\, x_i(\mathbf{D})\, x_m(\mathbf{D})\, \frac{1}{D^2}\,(f_2)_n \\ &= \mu^2 \langle \mathrm{v}_k\rangle_Y\, [(1+2g)\langle \mathrm{x}_i\rangle_Y]\, [(1+2g)\langle \mathrm{x}_m\rangle_Y]\, [(1+2g)\langle \mathrm{v}_n\rangle_Y]. \end{aligned} \tag{A.22}$$

Thus, to leading order the mean value of the product equals the product of the mean values:

$$\langle \mathrm{l}_i^2\rangle_Y = \langle \mathrm{l}_i\rangle_Y^2\, [1+\mathcal{O}(g)]\,. \tag{A.23}$$

3. Exact expressions

Exact mean values were already listed in Section IV for position and velocity in order to implement initial conditions. As shown above, the integrands for the mean values can be brought into the form (A.1), where $f_n(\mathbf{u})$ is a polynomial and in some cases has an additional factor 1/r. Each monomial of the polynomial is manipulated as follows:

$$\begin{aligned} u_1^k\, u_2^l\, u_3^m\, u_4^n P(\mathbf{u}) &= C_Y^2 \frac{\partial^{k+l+m+n}}{\partial a_1^k\, \partial a_2^l\, \partial a_3^m\, \partial a_4^n}\, |\psi_{\mathbf{a}}(\mathbf{u})|^2 \to C_Y^2\, \frac{\partial^{k+l+m+n}}{\partial a_1^k\, \partial a_2^l\, \partial a_3^m\, \partial a_4^n}\, F(\mathbf{a},\Gamma), & \text{(A.24)} \\ F(\mathbf{a},\Gamma) &= \int \mathrm{d}u_1 .... \mathrm{d}u_4\, |\psi_{\mathbf{a}}(\mathbf{u})|^2 = \frac{\pi^2}{\Gamma^2}\, \exp[A^2/(4\Gamma)]. & \text{(A.25)} \end{aligned}$$

In case, the factor 1/r is dealt with by means of parameter integration:

$$\frac{1}{\mathrm{r}}\, u_1^k\, u_2^l\, u_3^m\, u_4^n P(\mathbf{u}) \to C_Y^2 \int_\Gamma^\infty \mathrm{d}y\, \frac{\partial^{k+l+m+n}}{\partial a_1^k\, \partial a_2^l\, \partial a_3^m\, \partial a_4^n}\, F(\mathbf{a},y). \tag{A.26}$$

An uncompensated factor r is conveniently considered by the parameter differentiation

$$\mathrm{r}\, u_1^k\, u_2^l\, u_3^m\, u_4^n P(\mathbf{u}) \to C_Y^2 \left(-\frac{\partial}{\partial \Gamma}\right) \frac{\partial^{k+l+m+n}}{\partial a_1^k\, \partial a_2^l\, \partial a_3^m\, \partial a_4^n}\, F(\mathbf{a},\Gamma). \tag{A.27}$$

We used the computer algebra of Mathematica [13] to support and verify the manipulations.

The mean value of the angular momentum

$$\boldsymbol{L} = C_Y^2\, \mu \int \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \mathrm{d}u_4\, \mathrm{r}\, \psi^*\, (\mathbf{x}\times\mathbf{v})\, \psi. \tag{A.28}$$

can be written in the form

$$\boldsymbol{L} = \boldsymbol{X}\times\boldsymbol{P} + \boldsymbol{B}, \tag{A.29}$$

where we used the abbreviations

$$\boldsymbol{X} = \{\langle \mathrm{x}_1\rangle_O,\, \langle \mathrm{x}_2\rangle_O,\, \langle \mathrm{x}_3\rangle_O\}; \quad \boldsymbol{P} = \mu\, \{\langle \mathrm{v}_1\rangle_Y,\, \langle \mathrm{v}_2\rangle_Y,\, \langle \mathrm{v}_3\rangle_Y\}, \tag{A.30}$$

and

$$\begin{aligned} B_1 &= \mathbf{i}\, \frac{2\,\hbar}{A^2+8\Gamma}\, [a_1^* a_4 - a_1 a_4^* + a_2^* a_3 - a_2 a_3^*] \\ B_2 &= \mathbf{i}\, \frac{2\,\hbar}{A^2+8\Gamma}\, [a_1 a_3^* - a_1^* a_3 + a_2^* a_4 - a_2 a_4^*] \\ B_3 &= \mathbf{i}\, \frac{2\,\hbar}{A^2+8\Gamma}\, [a_1 a_2^* - a_1^* a_2 + a_3 a_4^* - a_3^* a_4]\,. \end{aligned} \tag{A.31}$$

We remark that the momentum and position operators obey the canonical commutation relations in u-space:

$$[\mu \mathrm{v}_i,\, \mathrm{x}_k]_u = \frac{\hbar}{\mathbf{i}}\, \delta_{ik}. \tag{A.32}$$

After assignment of the initial values according to (46) we find that the mean values are conserved in time:

$$L_1(\sigma) = L_2(\sigma) = 0; \quad L_3(\sigma) = \frac{1}{2}\, \hbar\, \nu\, (r_0 \Gamma))\, \frac{1+2g_0}{1+4g_0}. \tag{A.33}$$

The mean square deviation of x_1 at first can be brought into the following form

$$\begin{aligned} (\Delta x_1)^2 \equiv \langle x_1^2 \rangle_Y - (\langle x_1 \rangle_Y)^2 \;&=\; \frac{1}{2\Gamma^3 (A^2 + 8\Gamma)^2} \Big\{ -32\Gamma \left[(a_1 + a_1^*)(a_3 + a_3^*) - (a_2 + a_2^*)(a_4 + a_4^*) \right]^2 \\ &+\; A^2 (A^2 + 16\Gamma)^2 + 512\Gamma^3 \Big\}. \end{aligned} \tag{A.34}$$

After expressing the first part within the curly bracket in terms of $\langle x_1 \rangle_Y$, and introducing the abbreviation $g = 4\Gamma/A^2$, we can write

$$(\Delta x_1)^2 = \frac{2r_0}{\Gamma}\, \frac{1}{(r_0 \Gamma\, g)}\, \frac{1}{1+2g} \left[1 + 6g + 4g^2 \right] - \frac{4g^2}{(1+4g)^2}\, \langle x_1 \rangle_Y^2. \tag{A.35}$$

We assume that the eccentricity is sufficiently far away from 1, then $|x_1|$ is of the order r_0, and we can write

$$(\Delta x_1)^2 = \frac{2r_0^2}{r_0 \Gamma}\, \frac{1}{r_0 \Gamma\, g}\, [1 + \mathcal{O}(g)] = \frac{2r_0^2}{r_0 \Gamma}\, \frac{1 - e\cos(\Upsilon)}{1-e}\, [1 + \mathcal{O}(g)]\,. \tag{A.36}$$

Thus, if e is sufficiently far away from 1, we infer that

$$(\Delta x_1)^2 / r_0^2 \le \frac{1+e}{1-e}\, \frac{1}{r_0 \Gamma}\, [1 + \mathcal{O}(g)] \le \frac{1+e}{1-e}\, g_0\, [1 + \mathcal{O}(g)]\,. \tag{A.37}$$

It turns out that the expressions for the second and third component are analogous, one simply replaces above x_1 by x_i, i=1,2,3.

With the aid of the substitutions $4\Gamma/A^2 \to g$ and $\hbar^2 \Gamma^2/\mu^2 \to 16\,\omega^2$, the mean square deviation of the velocity components can be cast into the following form (needs some efforts):

$$(\Delta v_i)^2 \equiv \langle \mathrm{v}_i^2 \rangle_Y - \langle \mathrm{v}_i \rangle_Y^2 = \frac{2\omega^2\, g}{1+2g}\, [K_0 + K_1\, U_i + K_2\, U_4]\,, \quad i = 1,2,3 \tag{A.38}$$

with

$$K_0 \;=\; 4\, g\, [1 - \exp[-1/g]]\,, \tag{A.39}$$

$$K_1 \;=\; \frac{g}{1+2g}\, \{1 - 2g + (1+2g)\exp[-1/g]\}\,, \tag{A.40}$$

$$K_2 \;=\; 1 - g\, (1 - \exp[-1/g])\,, \tag{A.41}$$

$$U_1 \;=\; \frac{g^2}{\Gamma^2}\, [a_1 a_3 - a_1^* a_3^* - a_2 a_4 + a_2^* a_4^*]^2\,, \quad U_2 = \frac{g^2}{\Gamma^2}\, [a_2 a_3 - a_2^* a_3^* + a_1 a_4 - a_1^* a_4^*]^2\,,$$

$$U_3 \;=\; \frac{g^2}{4\Gamma^2} \left[a_1^2 - (a_1^*)^2 + a_2^2 - (a_2^*)^2 - a_3^2 + (a_3^*)^2 - a_4^2 + (a_4^*)^2 \right]^2; \tag{A.42}$$

$$U_4 \;=\; \frac{g}{\Gamma}\, (|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2). \tag{A.43}$$

We evaluate U_i and Z with the aid of the assignments (46) and the relation (43):

$$\begin{aligned} U_1 \;&=\; -(1+\nu^2)\, (g\, r_0 \Gamma)^2\, \frac{(1+2g_0)^2}{(1+4g_0)^2}\, \sin^2(2\omega\, \sigma), \\ U_2 \;&=\; -4\nu^2\, (g\, r_0 \Gamma)^2\, \frac{(1+2g_0)^2}{(1+4g_0)^2}\, \cos^2(2\omega\, \sigma), \quad U_3 = 0; \end{aligned} \tag{A.44}$$

$$U_4 \;=\; (1+\nu^2)\, (g\, r_0 \Gamma)\, \frac{1+2g_0}{1+4g_0}. \tag{A.45}$$

We assume that $g \approx g_0$ which is true, if the eccentricity is sufficiently far away from 1. Because $g\, r_0 \Gamma \approx 1$, the mean square deviations are dominated by the K_2 term and can be written as follows

$$(\Delta v_i)^2 = \frac{4\,\omega^2\, g}{1-e}\,(g\, r_0 \Gamma)\,[1+\mathcal{O}(g)] = \frac{4\omega^2}{r_0 \Gamma}\,\frac{1-e}{[1-e\cos(\Upsilon)]^2}\,[1+\mathcal{O}(g)], \quad i=1,2,3, \tag{A.46}$$

where we set $1+\nu^2 = 2/(1-e)$ according to (60) and made use of (72). Thus, $(\Delta v_i)^2$ is of the order $g_0 \sqcup$, which is consistent with lowest order estimate (A.16). The magnitude ω is proportional to the initial velocity v_0, more precisely

$$\omega^2 \equiv (-E)/(2\mu) = \frac{1}{4}\, v_0^2\, \frac{1-e}{1+e}. \tag{A.47}$$

This follows, up to terms of relative order g, from energy conservation $E = \mu\, v_0^2/2 - \kappa/r_0$ and the properties $r_0 = a(1-e)$ and (64). The uncertainty product $\Delta x_i\, \mu \Delta v_i$ as given in (90) may serve as a consistency check.

B. Probability density $P(x,y)$

The probability density to find the planet at the point (x,y) can be found from the integral

$$P(x,y) = C_Y^2 \int_{\mathbf{R}^4} \mathrm{d}\mathbf{u}\, u^2\, \delta(x - x_1(\mathbf{u}))\delta(y - x_2(\mathbf{u}))\, |\psi(\mathbf{u})|^2. \tag{B.1}$$

Obviously, $\int \mathrm{d}x\mathrm{d}y\, P(x,y) = 1$. We use the Fourier representation of the Dirac delta functions, and write

$$P(x,y) = \frac{1}{4\pi^2} \int \mathrm{d}k_1 \mathrm{d}k_2\, \exp\left[\mathbf{i}(k_1\, x + k_2\, y)\right]\, \tilde{P}(k_1,k_2), \tag{B.2}$$

$$\tilde{P}(k_1,k_2) = C_Y^2 \int_{\mathbf{R}^4} \mathrm{d}\mathbf{u}\, u^2\, \exp[\phi(\mathbf{u})], \quad \phi(\mathbf{u}) = \phi_1(\mathbf{u}) + \phi_2(\mathbf{u}), \tag{B.3}$$

$$\phi_1 = (\mathbf{a}+\mathbf{a}^*)\cdot\mathbf{u}; \quad \phi_2 = -\mathbf{i}\,(k_1\, x_1(\mathbf{u}) + k_2\, x_2(\mathbf{u})) - \Gamma u^2. \tag{B.4}$$

Because of the KS transformation (1), the phase ϕ_2 is a quadratic form in $\mathbf{u}$. Using summation convention it is expressed in terms of a symmetric matrix as follows

$$\phi_2 = u_i F_{ik} u_k, \tag{B.5}$$

where

$$\mathrm{F} = \begin{pmatrix} -\Gamma, & 0, & -\mathbf{i}k_1, & -\mathbf{i}k_2 \\ 0, & -\Gamma, & -\mathbf{i}k_2, & \mathbf{i}k_1 \\ -\mathbf{i}k_1, & -\mathbf{i}k_2, & -\Gamma, & 0 \\ -\mathbf{i}k_2, & \mathbf{i}k_1, & 0, & -\Gamma \end{pmatrix}. \tag{B.6}$$

This matrix is diagonalized by the following matrix G,

$$\mathrm{G} = \frac{1}{\sqrt{2}} \begin{pmatrix} k_2/k, & -k_1/k, & 0, & 1 \\ k_1/k, & k_2/k, & 1, & 0 \\ -k_2/k, & k_1/k, & 0, & 1 \\ -k_1/k, & -k_2/k, & 1, & 0 \end{pmatrix}, \quad k = \sqrt{k_1^2 + k_2^2}, \tag{B.7}$$

with the result

$$\left(\mathrm{GFG}^T\right)_{ik} = \lambda_i\, \delta_{ik}, \quad \lambda_1 = \lambda_2 = -\mathbf{i}k - \Gamma, \quad \lambda_3 = \lambda_4 = \mathbf{i}k - \Gamma, \tag{B.8}$$

where the superscript T denotes transposition. After the orthogonal transformation $\mathbf{u} \to \mathbf{U}$ with $\mathbf{u} = \mathrm{G}^T\mathbf{U}$ the phase appears in the form

$$\phi = \sum_{i=1}^{4} \left[c_i U_i + \lambda_i U_i^2\right] = \sum_{i=1}^{4} \left\{\lambda_i \left(U_i + \frac{c_i}{2\lambda_i}\right)^2 - \frac{c_i^2}{4\lambda_i}\right\}; \quad c_i = \sum_{k=1}^{4} G_{ik}(a_k + a_k^*). \tag{B.9}$$

In the integrand (B.3) the factor u^2 is generated by the differentiation $-\partial/\partial\Gamma$. After carrying out the U integration, we obtain

$$\begin{aligned}\tilde{P}(k_1,k_2) &= -C_Y^2 \frac{\partial}{\partial\Gamma}\frac{\pi^2}{k^2+\Gamma^2}\exp\left[-\sum_{i=1}^4 c_i^2/(4\lambda_i)\right] \\ &= \frac{\pi^2 C_Y^2}{(k^2+\Gamma^2)^2}\left[2\Gamma+(k^2+\Gamma^2)\sum_{i=1}^4 c_i^2/(4\lambda_i^2)\right]\exp\left[-\sum_{i=1}^4 c_i^2/(4\lambda_i)\right]. \qquad \text{(B.10)}\end{aligned}$$

We assign (46) to the parameters $\mathbf{a}+\mathbf{a}^*$, set $\nu^2=(1+e)/(1-e)$, and incorporate the normalization C_Y^2 from (26) by using the relation $A^2/(4\Gamma)=1/2\, r_0\Gamma\, Z$, see (43), (53) and (54). After straightforward simplifications we obtain

$$\tilde{P}(k_1,k_2) = \frac{8\,\Gamma^5}{(A^2+8\Gamma)(k^2+\Gamma^2)^2}\, M_0\, \exp\left[M_1+\mathbf{i}\, M_2\right]; \qquad \text{(B.11)}$$

$$M_0 = 1+\frac{r_0\Gamma}{4}\frac{(\Gamma^2-k^2)\,Z}{k^2+\Gamma^2}-\mathbf{i}\,\frac{r_0\Gamma^2}{2(k^2+\Gamma^2)}\left[(k_1M_3+k_2M_4)\right], \qquad \text{(B.12)}$$

$$M_1 = -\frac{r_0\Gamma}{2}\frac{k^2}{k^2+\Gamma^2}, \qquad \text{(B.13)}$$

$$M_2 = -\frac{r_0\Gamma^2}{2(k^2+\Gamma^2)}\,(k_1M_3+k_2M_4)\,, \qquad \text{(B.14)}$$

$$M_3 = \frac{2}{1-e}[-e+\cos(2\omega\,\sigma)]; \quad M_4 = 2\nu\sin(2\omega\,\sigma). \qquad \text{(B.15)}$$

The following approximation is appropriate. Since $P(x,y)$ will be peaked along the elliptic orbit, $|x|$ and $|y|$ are of the order r_0. As a consequence, in the Fourier integral (B.2) the main contribution will arise from the interval (k_1,k_2) with $0\le k_{1,2} <\approx 1/r_0$, and $k^2+\Gamma^2 <\approx 1/r_0^2+\Gamma^2=\Gamma^2[1+\mathcal{O}(g_0^2)]$. We therefore neglect k^2 as compared to Γ^2, and write

$$\tilde{P}(k_1,k_2) = N_0\,\exp\left[N_1+\mathbf{i}\,N_2\right]\,(1+\mathcal{O}(g_0)); \qquad \text{(B.16)}$$

$$N_0 = \frac{4}{r_0\Gamma\,Z}\left\{1+\frac{r_0\Gamma}{4}Z-\mathbf{i}\,\frac{r_0}{2}\left[k_1M_3+k_2M_4\right]\right\}, \qquad \text{(B.17)}$$

$$N_1 = -\frac{r_0\,Z}{2\Gamma}\,k^2, \qquad \text{(B.18)}$$

$$N_2 = -\frac{r_0}{2}\,\left[k_1M_3+k_2M_4\right]. \qquad \text{(B.19)}$$

To zero order with respect to g_0 one obtains

$$\tilde{P}(k_1,k_2) = \exp\left[N_1+\mathbf{i}\,N_2\right]\,(1+\mathcal{O}(g_0)). \qquad \text{(B.20)}$$

The remaining integration in (B.2) is elementary and leads to the result stated in (49).

We make the following observation: When compared with the mean position values, as given in (51), it is seen that

$$M_3 = \frac{2}{r_0}\,\langle x\rangle(1+\mathcal{O}(g_0)); \quad M_4 = \frac{2}{r_0}\,\langle y\rangle(1+\mathcal{O}(g_0)). \qquad \text{(B.21)}$$

C. Mean value of $\mathbf{H}_Y$

We state the following results for further evaluation:

$$\left\langle \frac{1}{\mathrm{r}}\,\Delta_u \right\rangle_Y = \frac{1}{1+2g}\frac{\Gamma^2}{A^2}\left[(a_1-a_1^*)^2+(a_2-a_2^*)^2+(a_3-a_3^*)^2+(a_4-a_4^*)^2-8\Gamma\right]; \qquad \text{(C.1)}$$

$$\left\langle \frac{1}{\mathrm{r}} \right\rangle_Y = \frac{1}{(1+2g)}\frac{4\Gamma^2}{A^2} = \frac{1}{S} \equiv \frac{\partial\sigma}{\partial t}; \qquad \text{(C.2)}$$

$$\left\langle \frac{\mathrm{Y}^2}{\mathrm{r}^2} \right\rangle_Y = \xi_1 \,(a_1^* a_2 - a_1 a_2^* - a_3^* a_4 + a_3 a_4^*)^2 + \xi_2 \, g^2 \varGamma (a_1 a_1^* + a_2 a_2^* + a_3 a_3^* + a_4 a_4^*);$$

$$\xi_1 = \frac{g^2}{4} \frac{1-2g+2g^2}{1+2g} \left[1 - \frac{2g^2}{1-2g+2g^2} \exp(-1/g) \right],$$

$$\xi_2 = -\frac{1}{2} \frac{1-g}{1+2g} \left[1 + \frac{g}{1-g} \exp(-1/g) \right]. \tag{C.3}$$

The assignments (46) with $a_k = M_k + \mathbf{i}\, N_k$ lead to the intermediary results

$$(a_1 - a_1^*)^2 + (a_2 - a_2^*)^2 + (a_3 - a_3^*)^2 + (a_4 - a_4^*)^2 = -4a\varGamma^2[1 - e\cos(2\omega\,\sigma)]; \tag{C.4}$$

$$\frac{A^2}{\varGamma^2} = 4a \, \frac{1+2g_0}{1+4g_0} \, [1 - e\cos(2\omega\,\sigma)]. \tag{C.5}$$

After replacing $\varGamma^2$ according to (23), we find the following time dependent mean values of the kinetic and potential energy

$$-\frac{\hbar^2}{8\mu} \left\langle \frac{1}{\mathrm{r}} \Delta_u \right\rangle_Y = \frac{1+4g_0}{1+2g_0} \, (-E) \, \frac{1}{1+2g(\sigma)} \, \frac{1 + e\cos(2\omega\,\sigma) + 2/(a\varGamma)}{1 - e\cos(2\omega\,\sigma)}; \tag{C.6}$$

$$-\kappa \left\langle \frac{1}{\mathrm{r}} \right\rangle_Y = -\frac{\kappa}{a} \, \frac{1}{1+2g(\sigma)} \, \frac{1}{1 - e\cos(2\omega\,\sigma)}, \tag{C.7}$$

which with the aid of (64) can be brought into the following form:

$$\left\langle -\frac{\hbar^2}{8\mu} \frac{1}{\mathrm{r}} \Delta_u - \frac{\kappa}{\mathrm{r}} \right\rangle_Y = \frac{E}{1+2g(\sigma)} \, \frac{1+4g_0}{1+2g_0} \left[1 - \frac{2/(a\varGamma)}{1 - e\cos(2\omega\,\sigma)} \right] = E\,\left[1 + \mathcal{O}(g_0)\right]. \tag{C.8}$$

The contribution of the constraint operator (C.3) is negligible: Because of

$$\left\langle [a_1^* a_2 - a_1 a_2^* - a_3^* a_4 + a_3 a_4^*]^2 \right\rangle_Y = 0, \quad \langle [a_1 a_1^* + a_2 a_2^* + a_3 a_3^* + a_4 a_4^*] \rangle_Y = 2(1+\nu^2)\rho_0^2, \tag{C.9}$$

we obtain with the aid of (43) and (C.9), after setting $g^2 = 16\varGamma^2/A^4$,

$$\frac{\hbar^2}{8\mu} \left\langle \frac{Y^2}{r^2} \right\rangle_Y = E\, g_0 \, \frac{1-e}{[1 - e\cos(\varUpsilon)]^2} \, \frac{1 - g + g\exp[-1/g]}{1 + 2\,g}. \tag{C.10}$$

Thus, if the eccentricity is not too close to 1, the relative contribution of the constraint to the energy is of the order g_0.

D. On Lenz vector

We give some comments to Nauenberg's paper [14]. In our notation, and without using special units, the Lenz vector reads:

$$M = \frac{1}{2\,\mu\kappa} \left[\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p} \right] - \frac{\mathbf{r}}{r}. \tag{E.1}$$

This vector has length e and points from the origin of the elliptic orbit towards perihelion. The commutator $[M_x, M_y]$, in restored units, has the form

$$[M_x, M_y] = 2\,\frac{\hbar}{\mathbf{i}}\, \frac{1}{\mu\kappa^2}\, H_x L_z. \tag{E.2}$$

The expectation value with the Gerry state schould be of order g_0, because, according to Appendix A.1, the expection value of an operator product $\langle AB \rangle = \langle A \rangle \langle B \rangle (1 + \mathcal{O}(g_0))$. As a matter of fact, we obtain with the aid of (65) and (A.33)

$$\langle [M_x, M_y] \rangle_Y = 2\,\frac{\hbar}{\mathbf{i}}\, \frac{1}{\mu\kappa^2}\, \langle H \rangle_Y \langle L_z \rangle_Y (1 + \mathcal{O}(g_0)) = \frac{\hbar^2}{\mathbf{i}}\, \frac{1}{\mu\kappa^2}\, (-E)\, \nu(r_0\varGamma)(1 + \mathcal{O}(g_0)). \tag{E.3}$$

Using (23) for Γ, $r_0 = (1+e)a$, $(-E) = \kappa/(2a)$, which implies $\kappa = 4a\mu\omega^2$, we can write omitting correction terms of order g_0

$$\langle [M_x, M_y] \rangle_Y = 2\,\frac{\hbar}{\mathbf{i}}\,\nu(1+e)\frac{\omega}{\kappa} = \frac{2}{\mathbf{i}}\,\nu(1+e)^2\,\frac{1}{r_0\Gamma} = \frac{2}{\mathbf{i}}\,\nu(1+e)^2\,g_0. \tag{E.4}$$

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